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The first conformal Dirac eigenvalue on 2-dimensional tori

Bernd Ammann*, Emmanuel Humbert

Institut Élie Cartan BP 239, Université Henri Poincaré, Nancy, 54506 Vandoeuvre-lès -Nancy, Cedex, France

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Abstract

Let M be a compact manifold with a spin structure χ and a Riemannian metric g. Let λ_g^2 be the smallest eigenvalue of the square of the Dirac operator with respect to g and χ. The τ-invariant is defined as

$$\tau(M, \chi) := \sup \inf \sqrt{\lambda_g^2} \mathrm{Vol}(M, g)^{1/n}$$

where the supremum runs over the set of all conformal classes on M, and where the infimum runs over all metrics in the given class.

We show that $\tau(T^2, \chi) = 2\sqrt{\pi}$ if χ is "the" non-trivial spin structure on T^2. In order to calculate this invariant, we study the infimum as a function on the spin-conformal moduli space and we show that the infimum converges to $2\sqrt{\pi}$ at one end of the spin-conformal moduli space.

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* Corresponding author. Tel.: +33 3 83 68 46 06; fax: +33 3 83 68 45 04.
E-mail addresses: ammann@iecn.u-nancy.fr (B. Ammann); humbert@iecn.u-nancy.fr (E. Humbert).

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1. Introduction

Let (M, g, χ) be a compact spin manifold of dimension $n \geq 2$. For any metric $\tilde{g}$ in the conformal class $[g]$ of g, let $\lambda_1(D_{\tilde{g}}^2)$ be the smallest eigenvalue of the square of the Dirac operator. We define

$$\lambda_{\min}(M, g, \chi) = \inf_{\tilde{g} \in [g]} \sqrt{\lambda_1(D_{\tilde{g}}^2)}\, \mathrm{Vol}(M, \tilde{g})^{1/n}.$$

Several works have been devoted to the study of this conformal invariant and some variants of it [17,22,8,2–4]. J. Lott [22,2] proved that $\lambda_{\min}(M, g, \chi) = 0$ if and only if $\ker D_g \neq \{0\}$. From [17,8] we deduce $\lambda_{\min}(\mathbb{S}^n) = \frac{n}{2}\omega_n^{1/n}$, where $\mathbb{S}^n$ is the sphere with constant sectional curvature 1 and where ω_n is its volume. Furthermore, in [2,6], we have seen that

$$\lambda_{\min}(M, g, \chi) \leq \lambda_{\min}(\mathbb{S}^n) = \frac{n}{2}\, \omega_n^{1/n} \tag{1}$$

for all Riemannian spin manifolds.

Furthermore, we define

$$\tau(M, \chi) := \sup \lambda_{\min}(M, g, \chi)$$

where the supremum runs over all conformal classes on M. Obviously, $\tau(M, \chi)$ is an invariant of a differentiable manifold with spin structure.

We consider it as interesting to determine τ or at least some bounds for τ in as many cases as possible. There are several motivations for studying these invariants $\lambda_{\min}(M, g, \chi)$ and $\tau(M, \chi)$.

Our first motivation is the analogy and the relation to Schoen's σ-constant, which is defined as

$$\sigma(M) := \sup \inf \frac{\int \mathrm{Scal}_g \, dv_g}{\mathrm{Vol}(M, g)^{(n-2)/n}}$$

where the infimum runs over all metrics in a conformal class $g \in [g_0]$, and where the supremum runs over all conformal classes.

In the case $\sigma(M) \geq 0$ and $n \geq 3$, there is also an alternative definition of the τ-invariant that is analogous to our definition of the τ-invariant. More exactly, in this case

$$\sigma(M) := \sup \inf \lambda_1(L_g) \mathrm{Vol}(M, g)^{2/n},$$

where $\lambda_1(L_g)$ is the first eigenvalue of the conformal Laplacian $L_g := 4(n-1)/(n-2)\Delta_g + \mathrm{Scal}_g$. Once again, the infimum runs over all metrics in a conformal class $g \in [g_0]$, and where the supremum runs over all conformal classes. Many conjectures about the value of the σ-constant exist, but unfortunately it can be calculated only in very few special cases, e.g. $\sigma(S^n) = n(n-1)\omega_n^{2/n}$, $\sigma(S^{n-1} \times S^1) = n(n-1)\omega_n^{2/n}$, $\sigma(T^n) = 0$ and $\sigma(\mathbb{R}P^3) = n(n-1)(\omega_n/2)^{2/n}$. The reader might consult [13] for a very elegant and amazing calculation of $\sigma(\mathbb{R}P^3)$ and for a good overview over further literature.

For other quotients of the sphere $\Gamma\backslash S^n$, $\Gamma \subset O(n+1)$ it is conjectured that

$$\sigma(\Gamma\backslash S^n) = n(n-1)\left(\frac{\omega_n}{\#\Gamma}\right)^{2/n}. \tag{2}$$

It is not difficult to show that for any metric conformal to the round metric on $\Gamma\backslash S^n$ one has the inequality $\lambda_1(L_g)\mathrm{Vol}(\Gamma\backslash S^n, g)^{2/n} \geq n(n-1)(\omega_n/\#\Gamma)^{2/n}$. This immediately implies $\sigma(\Gamma\backslash S^n) \geq n(n-1)(\omega_n/\#\Gamma)^{2/n}$, i.e. the lower bound on σ in (2). However, it is very difficult to obtain the upper bound on σ.

The τ-invariant is not only a formal analogue to Schoen's σ-constant, but it is also tightly related to it via Hijazi's inequality [17–19]. Hijazi's inequality implies that if M carries a spin structure χ, then

$$\tau(M,\chi)^2 \geq \frac{n}{4(n-1)}\sigma(M). \tag{3}$$

Equality is attained in this inequality if $M = S^n$. Hence, upper bounds for $\tau(M,\chi)$ may help to determine the σ-constant.

This is one reason for studying the τ-invariant.

Another motivation for studying $\tau(M,\chi)$ and $\lambda_{\min}(M,g,\chi)$ comes from the connection to constant mean curvature surfaces. Let $n = 2$. If $\tilde{g}$ is a minimizer that attains the infimum in the definition of $\lambda_{\min}(M,g,\chi)$, and if $\mathrm{Vol}(M,\tilde{g}) = 1$, then any simply connected open subset U of M can be isometrically embedded into $\mathbb{R}^3$, $(U,\tilde{g}) \hookrightarrow \mathbb{R}^3$, such that the resulting surface has constant mean curvature $\lambda_{\min}(M,g,\chi)$. Vice versa, any constant mean curvature surface gives rise to a stationary point of an associated variational principle. It is shown in [4] that minimizers of $\lambda_{\min}(M,g,\chi)$ exist if $\lambda_{\min}(M,g,\chi) < 2\sqrt{\pi}$.

Our third motivation comes from the search for metrics without harmonic spinors or with only few harmonic spinors. Let again $n \geq 2$ be arbitrary. As indicated above, $\lambda_{\min}(M,g,\chi) > 0$ if and only if $\ker D_g = \{0\}$. Hence, $\tau(M,\chi) > 0$ if and only if M carries a metric with $\ker D = \{0\}$. It follows from the Atiyah-Singer index theorem that any spin manifold M of dimension $4k$, $k \in \mathbb{N}$ with $\hat{A}(M) \neq 0$ has $\tau = 0$, and the same holds for spin manifolds of dimension $8k+1$ and $8k+2$ with non-vanishing α-genus. C. Bär conjectures [9,10] that in all remaining cases one has $\tau > 0$. Using perturbation methods Maier [23] has verified the conjecture in the case $n \leq 4$. The conjecture also holds if $n \geq 5$ and $\pi_1(M) = \{e\}$. Namely, if M is a compact simply connected spin manifold with vanishing α-genus, then building on Gromov–Lawson's surgery results [16] Stolz showed [27] that M carries a metric g_+ of positive scalar curvature. Applying the Schrödinger-Lichnerowicz formula we obtain $\ker D_{g_+} = \{0\}$, and hence $\tau(M,\chi) \geq \lambda_{\min}(M,g_+,\chi) > 0$ for the unique spin structure χ on M. A good reference for this argument is also [10], where the interested reader can also find an analogous statement for the case $\alpha(M,\chi) \neq 0$. The method of Stolz and Bär-Dahl also applies to some other fundamental groups, but the general case still remains open.

In the present article we want to have a closer look at the τ-invariant on surfaces, in particular 2-dimensional tori. The higher dimensional case will be the subject of another publication.

On surfaces the Yamabe operator cannot be defined as above. The Gauss-Bonnet theorem says that the σ-constant of a surface does not depend on the metric:

$$\sigma(M) = \sup\inf \int 2K_g\, dv_g = 4\pi\chi(M).$$

It was conjectured by Lott [22] and proved by C. Bär [8] that Eq. (3) also holds in dimension 2. This amounts in showing $\tau(S^2) = 2\sqrt{\pi}$. If M is a compact orientable surface of higher genus, then inequality (3) is trivial.

We will calulate the τ-invariant for the 2-dimensional torus T^2. The 2-dimensional torus T^2 has 4 different spin structures. The diffeomorphism group Diff(T^2) acts on the space of spin structures by pullback, and the action has two orbits: one orbit consisting of only one spin structure, the so-called *trivial spin structure* χ_{tr} and another orbit consisting of three spin structures. The torus T^2 equipped with the trivial spin structure has non-vanishing α-genus, thus $\tau = 0$. The main result of this article is the following theorem.

Theorem 1.1. *Let χ be a non-trivial spin structure on the 2-dimensional torus T^2. Then*

$$\tau(T^2, \chi) = 2\sqrt{\pi}(= \lambda_{\min}(\mathbb{S}^2)).$$

More exactly, for a fixed non-trivial spin structure χ we will study $\lambda_{\min}(M, g, \chi)$ as a function on the spin-conformal moduli space $\mathcal{M}$. We show that it is continuous (Proposition 3.1), and we show that it can be continuously extended to the natural 2-point compactification of $\mathcal{M}$, i.e. the compactification where both ends are compactified by one point each. It will be easy to show that $\lambda_{\min}(M, g, \chi) \to 0$ at one of the ends. However, it is much more involved to prove Theorem 3.2 which states that $\lambda_{\min}(M, g, \chi) \to \lambda_{\min}(\mathbb{S}^2) = 2\sqrt{\pi}$ at the other end.

It is evident that Theorem 3.2 implies Theorem 1.1.

For the Proof of Theorem 3.2, we have to establish a qualitative lower bound for the eigenvalues. One important ingredient in the Proof of Theorem 3.2 is to study a suitable covering of the 2-torus by a cylinder, and to lift a test spinor to this covering. Using a cut-off argument in a way similar to [5] we obtain a compactly supported test spinor on the cylinder. After compactifying the cylinder conformally to the sphere S^2, we can use Bär's 2-dimensional version of (3), to prove $\lambda_{\min}(M, g, \chi) \to \lambda_{\min}(\mathbb{S}^2) = 2\sqrt{\pi}$ at the other end.

Theorems 3.2 and 1.1 should be seen as a spinorial analogue of [26]. In that article, Schoen studies the Yamabe invariant on the moduli space of $O(n)$-invariant conformal structures on $S^1 \times S^{n-1}$, $n \geq 3$. He shows that at one end of this moduli space, the Yamabe invariant converges to the Yamabe invariant of $\mathbb{S}^n$, and hence $\sigma(S^1 \times S^{n-1}) = \sigma(\mathbb{S}^n)$. Combining this result with the Hijazi inequality and Theorem 1.1, one obtains

Corollary 1.2. *Let $n \geq 2$. Then*

$$\tau(S^{n-1} \times S^1, \chi) = \begin{cases} 0 & \textit{if } n = 2 \textit{ and if } \chi \textit{ is trivial}, \\ \dfrac{n}{2}\omega_n^{1/n} & \textit{otherwise} \end{cases}$$

The structure of the article is as follows.

In Section 2, we define the spin-conformal moduli $\mathcal{M}$ space of 2-tori and recall some well known facts. In Section 3, we state and explain our results. In Sections 4, we recall some preliminaries which will be useful for the Proof of Theorem 3.2. In Section 5 the proof is carried out.

2. The spin-conformal moduli space of T^2

At the beginning of this section we will recall the definition of a spin structure. We will only give it in the case $n = 2$. For more information and for the case of general dimension we refer to standard text books [15,21,25,12]. More details about the 2-dimensional case can be obtained in [5] and [1,11].

Let (M, g) be an oriented surface with a Riemannian metric g. Let $P_{\mathrm{SO}}(M, g)$ denote the set of oriented orthonormal frames over M. The base point map $P_{\mathrm{SO}}(M, g) \to M$ is an S^1 principal bundle. Let $\alpha : S^1 \to S^1$ be the non-trivial double covering, i.e. $\alpha(z) = z^2$. A *spin structure on* (M, g) is by definition a pair (P, χ) where P is an S^1 principal bundle over M and where $\chi : P \to P_{\mathrm{SO}}(M, g)$ is a double covering, such that the diagram

$$\begin{tikzcd} P \times S^1 \arrow[r] \arrow[dd, "\chi \times \alpha"'] & P \arrow[dd, "\chi"] \arrow[dr] & \\ & & M \\ P_{\mathrm{SO}}(M) \times S^1 \arrow[r] & P_{\mathrm{SO}}(M) \arrow[ur] & \end{tikzcd} \tag{4}$$

commutes (in this diagram the horizontal flashes denote the action of S^1 on P and $P_{\mathrm{SO}}(M)$). By slightly abusing the notation we will sometimes write χ for the spin structure, assuming that the domain P of χ is implicitly given. Two spin structures (P, χ) and $(\tilde{P}, \tilde{\chi})$ are isomorphic if there is an S^1-equivariant bijection $b : P \to \tilde{P}$ such that $\tilde{\chi} = \chi \circ b$.

If $\tilde{g} = f^2 g$ is a metric conformal to g. Then $P_{\mathrm{SO}}(M, \tilde{g}) \to P_{\mathrm{SO}}(M, g)$, $(e_1, e_2) \mapsto (fe_2, fe_2)$ defines an isomorphism of S^1 principal bundles. The pullback of a spin structure on (M, g) is a spin structure on $(M, \tilde{g})$.

In a similar way, if $(M_1, g_1) \to (M_2, g_2)$ is an orientation preserving conformal map, but not necessarily a diffeomorphism, then any spin structure on (M_2, g_2) pulls back to a spin structure on (M_1, g_1).

Examples 2.1.

(1) If g_0 is the standard metric on S^2. Then $P_{\mathrm{SO}}(S^2, g_0) = \mathrm{SO}(3)$, and the base point map $\mathrm{SO}(3) \to S^2$ is the map that associates to a matrix in SO(3) the first column. The double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$ defines a spin structure on (S, g_0).

(2) Let $\tilde{g}$ be an arbitrary metric on S^2. After a possible pullback by a diffeomorphism $S^2 \to S^2$ we can write $\tilde{g} = f^2 g_0$. The pullback of the spin structure given in (1) under the isomorphism $P_{\mathrm{SO}}(S^2, \tilde{g}) \to P_{\mathrm{SO}}(S^2, g_0)$ defines a spin structure on $(S^2, \tilde{g})$.

(3) Let g_1 be a flat metric on the torus T^2. Then a parallel frame gives rise to a (global) section of $P_{\mathrm{SO}}(T^2, g_1) \to T^2$. Hence, this is a trivial S^1 principal bundle. The trivial fiberwise double covering $T^2 \times S^1 \to T^2 \times S^1$, $(p, z) \mapsto (p, z^2)$ defines a spin structure on (T^2, g_1), the so-called *trivial spin structure* χ_{tr} on (T^2, g_1).

(4) If $\tilde{g}$ is an arbitrary metric on T^2. Then we can write $\tilde{g} = f^2 g_1$ where g_1 is a flat metric. As above, the trivial spin stucture on (T^2, g_1) defines a spin structure on $(T^2, \tilde{g})$. This spin structure is also called the *trivial spin structure* χ_{tr}.

(5) For $(x_0, y_0) \in \mathbb{R}^2 \setminus \{0\}$ we define

$$Z_{x_0,y_0} = \mathbb{R}^2/\langle (x_0, y_0)\rangle$$

where $\langle (x_0, y_0)\rangle$ is the subgroup of $\mathbb{R}^2$ spanned by (x_0, y_0). We will assume that it carries the metric induced by the euclidean metric g_{eucl} on $\mathbb{R}^2$. Then $P_{\mathrm{SO}}(Z_{x_0,y_0})$ is a trivial bundle, and a natural trivialization is obtained by a parallel frame. The map $Z_{x_0,y_0} \times S^1 \to Z_{x_0,y_0} \times S^1$, $(p, z) \mapsto (p, z^2)$ defines a spin structure on Z_{x_0,y_0}, the *trivial spin structure* on Z_{x_0,y_0}.

Assume that $\chi : P \to P_{\mathrm{SO}}(M, g)$ is a spin structure on a surface, and assume that $\beta : \pi_1(M) \to \{-1, +1\}$ is a group homomorphism. Then there is a $\{-1, +1\}$ principal bundle $B_\beta \to M$ with holonomy β. Let P_β be the quotient of $P \times B_\beta$ by the diagonal action of $\{-1, +1\}$. Then P_β together with the induced map $\chi_\beta : P_\beta \to P_{\mathrm{SO}}(M, g)$ is also a spin structure on (M, g). Conversely, if $(\tilde{P}, \tilde{\chi})$ is another spin structure, then one can show that there is a unique $\beta : \pi_1(M) \to \{-1, +1\}$ such that $(\tilde{P}, \tilde{\chi})$ and (P_β, χ_β) are isomorphic. Thus, we see that the space of spin structures is an affine space over the $\{-1, +1\}$-vector space $\mathrm{Hom}(\pi_1(M), \{-1, +1\}) = H^1(M, \{-1, +1\})$.

Examples 2.2.

(1) Any compact oriented surface M carries a spin structure. If k denotes the genus of M, then there are 4^k homomorphisms $\pi_1(M) \to \{-1, +1\}$, hence there are 4^k isomorphism classes of spin structures. In particular, the spin structure on S^2 is unique.
(2) Because of $\pi_1(Z_{x_0,y_0}) = \mathbb{Z}$, there are exactly two spin structures on Z_{x_0,y_0}, the trivial one and another one called the *non-trivial spin structure*.

From now on, let $M = T^2 = \mathbb{R}^2/\Gamma$ where Γ is a lattice in $\mathbb{R}^2$. The trivial spin structure defined above can be used to identify $\mathrm{Hom}(\pi_1(M), \{-1, +1\})$ with the set of isomorphism classes of spin structures. By slightly abusing the language we will always write χ for the spin structure (P, χ) and also for the homomorphism $\pi_1(M) \to \{-1, +1\}$.

The following lemma summarizes some well-known equivalent characterizations of triviality of χ (see e.g. [21,24,1,14]).

Lemma 2.3. *With the above notations, the following statements are equivalent*

(1) *The spin structure is trivial* (*in the above sense*);
(2) $\chi(\gamma) = 1$ *for all* $\gamma \in \Gamma$;
(3) *The spin structure is invariant under the natural action of the diffeomorphism group* $\mathrm{Diff}(T^2)$;
(4) (T^2, χ) *is the* **non**-*trivial element in the 2-dimensional spin-cobordism group*;
(5) *The* α-*genus of* (T^2, χ) *is the non-trivial element in* $\mathbb{Z}/2\mathbb{Z}$;
(6) *The Dirac operator has a non-trivial kernel*;
(7) *The kernel of the Dirac operator has complex dimension 2*.

In particular, we easily see

$$\tau(T^2, \chi_{\mathrm{tr}}) = 0.$$

From now on, in the rest of this article, we assume that χ is not the trivial spin structure, i.e. $\chi(\gamma) = -1$ for some $\gamma \in \Gamma$.

Definition 2.4. Two 2-dimensional tori with Riemannian metrics, orientations and spin structures are said to be *spin-conformal* if there is a conformal map between them preserving the orientation and the spin structure. “Being spin-conformal” is obviously an equivalence relation. The *spin-conformal moduli space $\mathcal{M}$ of T^2 with the non-trivial spin structure* is defined to be the set of these equivalence classes. Furthermore we define

$$\mathcal{M}_1 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| |x| \le \frac{1}{2}, \quad y^2 + \left(|x| - \frac{1}{2}\right)^2 \ge \frac{1}{4}, \quad y > 0 \right\}$$

(see also Fig. 1).

Lemma 2.5. *Let g be a Riemannian metric on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, and let $\chi : \mathbb{Z}^2 \to \{-1, +1\}$ be a non-trivial spin structure. Then there is a lattice $\Gamma \subset \mathbb{R}^2$, a spin structure $\chi' : \Gamma \to \{-1, +1\}$, such that*

(1) Γ *is generated by* $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} x \\ y \end{pmatrix}$ *with* $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{M}_1$
(2) (T^2, g, χ) *is spin-conformal to* $(\mathbb{R}^2/\Gamma, g_{\mathrm{eucl}}, \chi')$
(3) $\chi'\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = +1$ *and* $\chi'\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = -1.$

Proof. Because of the uniformization theorem we can assume without loss of generality that g is a flat metric. The lemma then follows from elementary algebraic arguments. $\square$

Note that x and y are uniquely determined if $\begin{pmatrix} x \\ y \end{pmatrix}$ is in the interior of $\mathcal{M}_1$, i.e. if $|x| < 1/2$ and $y^2 + (|x| - 1/2)^2 > 1/4$. If $\begin{pmatrix} x \\ y \end{pmatrix}$ is on the boundary of $\mathcal{M}_1$, then y and $|x|$ are determined, but not the sign of x. Hence, after gluing $\begin{pmatrix} x \\ y \end{pmatrix} \in \partial\mathcal{M}_1$ with $\begin{pmatrix} -x \\ y \end{pmatrix}$ we obtain the spin-conformal moduli space $\mathcal{M}$.

Notation. Let $(x_0, y_0) \in \mathcal{M}_1$. The lattice generated by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix}$ is noted as Γ_{x_0,y_0}. Furthermore, we write T_{x_0,y_0} for the 2-dimensional torus $\mathbb{R}^2/\Gamma_{x_0,y_0}$ *equipped with the euclidean metric.*

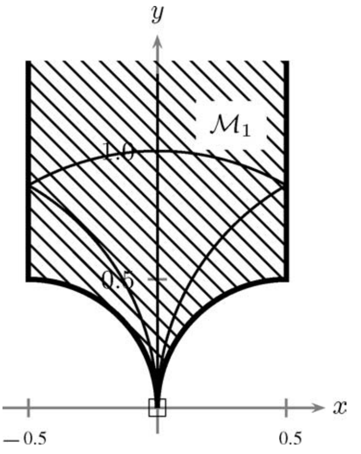

Fig. 1. The spin conformal moduli space is $\mathcal{M} = \mathcal{M}_1/\sim$, where $\sim$ means identifying $(x, y) \in \partial\mathcal{M}_1$ with $(-x, y)$.

The quantity $\lambda_{\min}(T^2, g, \sigma)$ is a spin-conformal invariant, hence $\lambda_{\min}$ can be viewed as a function on $\mathcal{M}$ or on $\mathcal{M}_1$.

3. Main results

In this article, we study $\lambda_{\min}$ as a function on the spin-conformal moduli space with the non-trivial spin structure. This function takes values in $[0, \lambda_{\min}(\mathbb{S}^2)]$ because of (1). As the spin structure is non-trivial, Lott's results states that 0 is not attained. As a preliminary result we will prove that this function is continuous.

Proposition 3.1. *The function*

$$\lambda_{\min} : \left| \begin{array}{ll} \mathcal{M}_1 & \to\ ]0, \lambda_{\min}(\mathbb{S}^2)] \\ (x_0, y_0) & \mapsto \lambda_{\min}^{x_0, y_0} \end{array} \right.$$

is continuous on $\mathcal{M}_1$.

The spin-conformal moduli space $\mathcal{M}$ (resp. $\mathcal{M}_1$) has two ends. We will compactify each end by adding one point. The point added at the end $y \to \infty$ will be denoted by ∞ and the point added at the end $y \to 0$ is denoted by $(0, 0)$.

Theorem 3.2. *The function*

$$\lambda_{\min} : \left| \begin{array}{ll} \mathcal{M}_1 & \to\ ]0, \lambda_{\min}(\mathbb{S}^2)] \\ (x_0, y_0) & \mapsto \lambda_{\min}^{x_0, y_0} \end{array} \right.$$

extends continuously to $\mathcal{M}_1 \cup \{(0,0), \infty\}$ *by setting* $\lambda_{\min}^{0,0} = \lambda_{\min}(\mathbb{S}^2)$ *and* $\lambda_{\min}^{\infty} = 0$.

The continuous extension at ∞ is is easy to see. The first eigenvalue of the Dirac operator on $(T_{x_0,y_0}, g_{\mathrm{eucl}}, \chi_{x_0,y_0})$, is π/y_0, the area is y_0, hence

$$\lambda_{\min}^{x_0,y_0} \le \pi/\sqrt{y_0} \to 0 \qquad \text{for } y_0 \to \infty.$$

However, the limit $(x_0, y_0) \to (0, 0)$ is much more difficult to obtain.

Clearly, Theorem 3.2 implies Theorem 1.1.

4. Some preliminaries

4.1. Variational characterization of $\lambda_{\min}$

Let (M, g, χ) be a compact spin manifold of dimension $n \ge 2$ with $\ker D_g = \{0\}$. For $\psi \in \Gamma(\Sigma M)$, we define

$$J_g(\psi) = \frac{(\int_M |D\psi|^{2n/n+1} dv_g)^{n+1/n}}{|\int_M \langle D\psi, \psi\rangle dv_g|}.$$

Lott [22] proved that

$$\lambda_{\min}(M, [g], \chi) = \inf_{\psi} J_g(\psi) \tag{5}$$

where the infimum is taken over the set of smooth spinor fields for which

$$\left(\int_M \langle D\psi, \psi\rangle dv_g\right) \ne 0.$$

The functional J_g for the torus T_{x_0,y_0} is noted as J^{x_0,y_0}.

Remark 4.1. The exponents in J_g are chosen such that J_g is conformally invariant. More exactly, if g and $\tilde{g}$ are conformal, then the spinor bundles of (M, g, χ) and $(M, \tilde{g}, \chi)$ can be identified in such a way that $J_g(\psi) = J_{\tilde{g}}(\psi)$.

4.2. Cylinders and doubly pointed spheres

Let Z_{x_0,y_0} be defined as in Examples 2.1 (5).

Lemma 4.2 (Mercator, around 1569). *Let* $N, S \in \mathbb{S}^2$ *be respectively the North pole and the South pole of* $\mathbb{S}^2$. *Then there is a conformal diffeomorphism* F_{x_0,y_0} *from* $(Z_{x_0,y_0}, g_{\mathrm{eucl}})$ *to* $(\mathbb{S}^2 \setminus \{N, S\})$.

Proof. In the case $(x_0, y_0) = (0, 2\pi)$ we see that the application

$$F_{0,2\pi} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{\sin y}{\cosh x} \\ \frac{\cos y}{\cosh x} \\ \tanh x \end{pmatrix}$$

is conformal and defines a conformal bijection $Z_{0,2\pi} \to S^2 \setminus \{N, S\}$. The general case follows by composing with a linear conformal map $Z_{x_0,y_0} \to Z_{0,2\pi}$. □

The map F induces a map between the frame bundles.

$$\begin{aligned} &\tilde{F}_{x_0,y_0} : P_{\mathrm{SO}}(Z_{x_0,y_0}) \to P_{\mathrm{SO}}(\mathbb{S}^2) \\ &\tilde{F}_{x_0,y_0}((p, X, Y)) := \left(F_{x_0,y_0}(p), \frac{dF_{x_0,y_0}(X)}{|dF_{x_0,y_0}(X)|}, \frac{dF_{x_0,y_0}(Y)}{|dF_{x_0,y_0}(Y)} \right) \\ &X, Y \in T_p Z_{x_0,y_0} \text{ are orthonormal and oriented} \end{aligned}$$

The unique spin structure on $\mathbb{S}^2$ pulls back to a spin structure on Z_{x_0,y_0}, that we will denote as χ_{x_0,y_0}.

Lemma 4.3. *The spin structure χ_{x_0,y_0} is the* ***non-trivial*** *spin structure on Z_{x_0,y_0}.*

Proof. We will show the lemma for the case $(x_0, y_0) = (0, 2\pi)$. As before, the general case then follows by composing with a linear map $Z_{x_0,y_0} \to Z_{0,2\pi}$.

We define the loop $\gamma : [0, 2\pi] \to Z_{0,2\pi}$, $\gamma(t) := (0, t)$ and the parallel section

$$\alpha : t \mapsto \left(\frac{\partial}{\partial x}|_{\gamma(t)}, \frac{\partial}{\partial y}|_{\gamma(t)} \right)$$

of $P_{\mathrm{SO}}(Z_{0,2\pi})$ along γ. The spin structure $(P, \chi_{0,2\pi})$ on $Z_{0,2\pi}$ is trivial if and only if there is a section $\tilde{\alpha}$ of P along γ such that $\chi_{0,2\pi} \circ \tilde{\alpha} = \alpha$ and $\tilde{\alpha}(0) = \tilde{\alpha}(2\pi)$.

The composition $\tilde{F}_{0,2\pi} \circ \alpha$ is a section of $P_{\mathrm{SO}}(\mathbb{S}^2) = \mathrm{SO}(3)$ along $F_{0,2\pi} \circ \gamma$. One checks that

$$\tilde{F}_{0,2\pi} \circ \alpha(t) = \left(\frac{\partial F_{0,2\pi}}{\partial x}|_{(0,t)}, \frac{\partial F_{0,2\pi}}{\partial y}|_{(0,t)}, F(0, t) \right) = \begin{pmatrix} 0 & \cos y & \sin y \\ 0 & -\sin y & \cos y \\ 1 & 0 & 0 \end{pmatrix}$$

We lift this loop to a path $\hat{\alpha}$ in SU(2), then one easily sees that $\hat{\alpha}(0) = -\hat{\alpha}(2\pi)$. As $\chi_{0,2\pi}$ is defined as the pullback of the spin structure on S^2, we see that any lift $\tilde{\alpha}$ of α also satisfies $\tilde{\alpha}(0) \neq \tilde{\alpha}(2\pi)$. Hence, we have proved non-triviality of $\chi_{0,2\pi}$. □

Corollary 4.4. *Let Z_{x_0,y_0} carry its non-trivial spin structure. Then,*

$$\frac{\left(\int_{Z_{x_0,y_0}} |D\psi|^{4/3} dx\right)^{3/2}}{|\int_{Z_{x_0,y_0}} \langle \psi, D\psi \rangle dx|} \geq \lambda_{\min}(\mathbb{S}^2)$$

for any compactly supported spinor $\psi \in \Gamma(\Sigma Z_{x_0,y_0})$ *such that* $\int \langle \psi, D\psi \rangle \neq 0$.

Let $f : Z_{x_0,y_0} \to]0, +\infty[$ be such that $F^*_{x_0,y_0} g_0 = f^2 g_{\mathrm{eucl}}$. It is well known (see for example [20,17]) that F_{x_0,y_0} induces a pointwise isometry

$$\left| \begin{array}{lll} \Sigma(T_{x_0,y_0}, g_{\mathrm{eucl}}) & \to & \Sigma(\mathbb{S}^2 \setminus \{N, S\}, g_0) \\ \psi & \mapsto & \overline{\psi}, \end{array} \right.$$

such that

$$\overline{D} f^{-(1/2)} \overline{\psi} = f^{-(3/2)} \overline{D\psi}.$$

where $\overline{D}$ denotes the Dirac operator on $\mathbb{S}^2$. Moreover, $\overline{\psi}$ is smooth on $\mathbb{S}^2$ since $\psi \equiv 0$ in a neighborhood of N and S. It is well known that the functional J defined at the beginning of Section 4 is conformally invariant. This implies that

$$\frac{\left(\int_{Z_{x_0,y_0}} |D\psi|^{4/3}\, dx\right)^{3/2}}{|\int_{Z_{x_0,y_0}} \langle \psi, D\psi \rangle dx|} = \frac{\left(\int_{\mathbb{S}^2} |\overline{D}(f^{-(1/2)}\overline{\psi})|^{4/3} dv_{g_0}\right)^{3/2}}{|\int_{\mathbb{S}^2} \langle f^{-(1/2)}\overline{\psi}, \overline{D}(f^{-(1/2)}\overline{\psi})\rangle dv_{g_0}|} \geq \lambda_{\min}(\mathbb{S}^2).$$

5. Proof of the main results

For the proof we will need the following well known elliptic estimates. These estimates are a consequence of techniques explained for example in [28], see also [7]. However, in our special situation a proof is much easier. Hence, for the convenience of the reader we will include an elementary proof here.

Lemma 5.1 (Elliptic estimates). *Let* $(x_0, y_0) \in \mathcal{M}_1$, *and note* T^2 *for* T_{x_0,y_0}. *There exists* $C > 0$ *depending only on* x_0 *and* y_0 *such that*

$$\int_{T^2} |D\psi|^{4/3} dv_g \geq C \int_{T^2} |\nabla\psi|^{4/3} dv_g \tag{6}$$

and

$$\left(\int_{T^2} |\psi|^4 dv_g\right)^{1/3} \leq C \int_{T^2} |\nabla\psi|^{4/3} dv_g \tag{7}$$

for any smooth spinor ψ.

Proof. Let $q = \frac{4}{3}$. Assume that (6) is false. Then, for all $\varepsilon > 0$, we can find a smooth spinor $\psi_\varepsilon \in \Gamma(\Sigma(T^2))$ such that

$$\int_{T^2} |D\psi_\varepsilon|^q dv_g \leq \varepsilon \qquad \text{and} \qquad \int_{T^2} |\nabla\psi_\varepsilon|^q dv_g = 1. \tag{8}$$

Now, assume that

$$\lim_{\varepsilon\to 0}\left(\int_T |\psi_\varepsilon|^q \, dv_g\right)^{1/q} = +\infty.$$

Then, we set

$$\psi'_\varepsilon = \frac{\psi_\varepsilon}{(\int_{T^2} |\psi_\varepsilon|^q dv_g)^{1/q}}.$$

The sequence (ψ'_ε) is bounded in $W^{1,q}(T^2)$ and since $W^{1,q}(T^2)$ is reflexive, we can find $\psi'_0 \in W^{1,q}(T^2)$ such that there is sequence $\varepsilon_i \to 0$, with $\lim_{i\to\infty} \psi'_{\varepsilon_i} = \psi'_0$ weakly in $W^{1,q}(T^2)$. Then, we would have

$$\int_{T^2} |\nabla \psi'_0|^q dv_g \le \liminf_{\varepsilon} \int_{T^2} |\nabla \psi'_\varepsilon|^q dv_g = 0$$

We would get that ψ'_0 is parallel which cannot occur since the structure on T^2 is not trivial. This proves that (ψ_ε) is bounded in $L^q(T^2)$ and hence, by (8) in $W^{1,q}(T^2)$. Again by reflexivity of $W^{1,q}(T^2)$, we get the existence of a spinor ψ_0, weak limit of a subsequence ψ_{ε_i} in $W^{1,q}(T^2)$. By weak convergence of $D\psi_{\varepsilon_i}$ to $D\psi_0$ in $L^q(T^2)$, we have

$$\int_{T^2} |D\psi_0|^q dv_g \le \liminf_{i} \int_{T^2} |D\psi_{\varepsilon_i}|^q dv_g = 0.$$

This is impossible since the Dirac operator on T^2 has a trivial kernel. This proves (6). As one can check, relation (7) can be proved with the same type of arguments. □

Proof of Proposition 3.1. The proposition states that $\lambda_{\min}$ is continuous on $\mathcal{M}_1$. Let $(x_k, y_k)_k \in \mathcal{M}_1$ be a sequence tending to $(x_0, y_0) \in \mathcal{M}_1$. We identify T^2 with $\mathbb{R}^2/\mathbb{Z}^2$. The conformal structures corresponding to (x_k, y_k) and (x_0, y_0) are represented by flat metrics g_{x_k,y_k} and g_{x_0,y_0} on $\mathbb{R}^2/\mathbb{Z}^2$, that are invariant under translations, and such that $g_{x_k,y_k} \to g_{x_0,y_0}$ in the C^∞-topology.

Let $\varepsilon > 0$ be small and let ψ_0 and $(\psi_k)_k$ be smooth spinors such that

$$J_{x_0,y_0}(\psi_0) \le \lambda_{\min}^{x_0,y_0} + \varepsilon \text{ and } J_{x_k,y_k}(\psi_k) \le \lambda_{\min}^{x_k,y_k} + \varepsilon.$$

At first, since $(g_{x_k,y_k})_k$ tends to g_{x_0,y_0}, it is easy to see that

$$\lim_k J_{x_k,y_k}(\psi_0) = J_{x_0,y_0}(\psi_0)$$

and hence $\limsup_k \lambda_{\min}^{x_k,y_k} \le \lambda_{\min}^{x_0,y_0} + \varepsilon$ for the given $\varepsilon > 0$ that we can choose as small as we want. Thus

$$\limsup_k \lambda_{\min}^{x_k,y_k} \le \lambda_{\min}^{x_0,y_0}.$$

Now, let us prove that

$$\limsup_k J_{x_0,y_0}(\psi_k) \le \liminf_k J_{x_k,y_k}(\psi_k) \tag{9}$$

We let (v, w) be a orthormal basis for g_{x_0,y_0} and $(v_k, w_k)_k$, orthonormal basis for g_{x_k,y_k} which tends to (v, w). One can write for all k, $v_k = a_k v + b_k w$ and $w_k = c_k v + d_k w$ with $\lim_k a_k = \lim_k d_k = 1$ and $\lim_k b_k = \lim_k c_k = 0$. We have

$$\begin{aligned}\left(\int_{T^2} |D_{x_k,y_k}\psi_k|^{4/3} dv_{g_{x_k,y_k}}\right)^{3/4} &= \left(\int_{T^2} |v_k\nabla_{v_k}\psi_k + w_k\nabla_{w_k}\psi_k|^{4/3} dv_{g_{x_k,y_k}}\right)^{3/4}\\ &= \left(\int_{T^2} |D_{x_0,y_0}\psi_k + \theta_k|^{4/3} dv_{g_{x_k,y_k}}\right)^{3/4}.\end{aligned}$$

with

$$\begin{aligned}|\theta_k| = {}&|(a_k^2 + c_k^2 - 1)v\nabla_v\psi_k + (b_k^2 + d_k^2 - 1)w\nabla_w\psi_k + (a_k b_k + c_k d_k)\\ &\times\langle w\nabla_v\psi_k + v\nabla_w\psi_k\rangle| \le \alpha_k|\nabla\psi_k|\end{aligned}$$

where $(\alpha_k)_k$ is a sequence of positive numbers which tends to 0. Note that because of the translation invariance of the metrics, the Levi-Civita connection does not depenpend on k. Since $\lim_k g_{x_k,y_x} = g_{x_0,y_0}$, one gets that

$$\begin{aligned}\left(\int_{T^2} |D_{x_k,y_k}\psi_k|^{4/3} dv_{g_{x_k,y_k}}\right)^{3/4} \ge{}& \left(\int_{T^2} |D_{x_0,y_0}\psi_k|^{4/3} dv_{g_{x_0,y_0}}\right)^{3/4}\\ &-\alpha'_k\left(\int_{T^2} |\nabla\psi_k|^{4/3} dv_{g_{x_0,y_0}}\right)^{3/4}\end{aligned}$$

where $\lim_k \alpha'_k = 0$. Together with Lemma 5.1, we get that

$$(1 - C\alpha'_k)\left(\int_{T^2} |D_{x_0,y_0}\psi_k|^{4/3} dv_{g_{x_0,y_0}}\right)^{3/4} \le \left(\int_{T^2} |D_{x_k,y_k}\psi_k|^{4/3} dv_{g_{x_k,y_k}}\right)^{3/4} \tag{10}$$

where C is a positive constant independent of k. Now, in the same way, we can write

$$\int_{T^2} \langle\psi_k, D_{x_0,y_0}\psi_k\rangle dv_{g_{x_0,y_0}} \ge \int_{T^2} \langle\psi_k, D_{x_k,y_k}\psi_k\rangle dv_{g_{x_k,y_k}} - \beta_k \int_{T^2} |\psi_k||\nabla\psi_k| dv_{g_{x_0,y_0}}$$

where $\lim_k \beta_k = 0$. Using Hölder inequality, we have

$$\int_{T^2} |\psi_k||\nabla\psi_k| dv_{g_{x_0,y_0}} \le \left(\int_{T^2} |\psi_k|^4 dv_{g_{x_0,y_0}}\right)^{1/4}\left(\int_{T^2} |\nabla\psi_k|^{4/3} dv_{g_{x_0,y_0}}\right)^{3/4}.$$

Using (6) and (7), this gives

$$\int_{T^2} |\psi_k||\nabla\psi_k| dv_{g_{x_0,y_0}} \leq C \left(\int_{T^2} |D\psi_k|^{4/3} dv_{g_{x_0,y_0}}\right)^{3/2}.$$

We obtain

$$\int_{T^2} \langle \psi_k, D_{x_0,y_0}\psi_k\rangle dv_{g_{x_0,y_0}}$$
$$\geq \int_{T^2} \langle \psi_k, D_{x_k,y_k}\psi_k\rangle dv_{g_{x_k,y_k}} - \beta_k \left(\int_{T^2} |D\psi_k|^{4/3} dv_{g_{x_0,y_0}}\right)^{3/2}.$$

Together with (10), we get (9). This immediatly implies that

$$\liminf_k \lambda_{\min}^{x_k,y_k} \geq \lambda_{\min}^{x_0,y_0}$$

and ends the proof of the proposition. □

Proof of Theorem 3.2. Any calculation in this proof will be carried out in Riemannian normal coordinates with respect to a flat metric. In the following, (e_1, e_2) will denote the canonical basis of $\mathbb{R}^2$.

In order to prove $\lim_{(x_0,y_0)\to(0,0)} \lambda_{\min}^{x_0,y_0} = \lambda_{\min}(\mathbb{S}^2)$ we will show that there is no sequence $(x_k, y_k) \to (0, 0)$ such that $\lim_{(x_k,y_k)\to(0,0)} \lambda_{\min}^{x_k,y_k} < \lambda_{\min}(\mathbb{S}^2)$. We may assume that $\lambda_{\min}^{x_k,y_k} < \lambda_{\min}(\mathbb{S}^2)$ for all k. Note that the spectrum of D is symmetric in dimension 2. By [3], we then can find a sequence of spinors ψ_k of class C^1 such that on T_{x_k,y_k}

$$D\psi_k = \lambda_{\min}^{x_k,y_k} |\psi_k|^2 \psi_k \tag{11}$$

and such that

$$\int_{T_{x_k,y_k}} |\psi_k|^4 dx = 1. \tag{12}$$

Moreover, we have

$$J_{x_k,y_k}(\psi_k) = \lambda_{\min}^{x_k,y_k}. \tag{13}$$

Sometimes we will identify ψ_k with its pullback to $\mathbb{R}^2$. In this picture ψ_k is a doubly periodic spinor on $\mathbb{R}^2$.

Step 1. There exists $C > 0$ such that for all k, we have $\lambda_{\min}^{x_k,y_k} \geq C y_k^{1/2}$.

Here and in the sequel, C will always denote a positive constant which does not depend on k.

For the proof of the first step, we let $\Omega = \{(x, y) \in \mathcal{M}_1 \mid 1/2 \leq y \leq 3/2\}$. Since Ω is compact and since $\lambda_{\min}$ is continuous and positive, there exists $C > 0$ such that for all

$$\lambda_{\min} \geq C \text{ on } \Omega. \tag{14}$$

Now, assume that

$$\lim_k \frac{\lambda_{\min}^{x_k,y_k}}{y_k^{1/2}} = 0.$$

We can find a sequence $(N_k)_k$ which tends to $+\infty$ such that $(3^{N_k}x_k, 3^{N_k}y_k) \in \Omega$. Note that the locally isometric covering $T_{px_k,py_k} \to T_{x_k,y_k}$, $p \in \mathbb{N}$, preserves the spin structures if and only if p is odd. Let $\tilde{\psi}_k$ be the pullback of ψ_k with respect to covering $T_{3^{N_k}x_k,3^{N_k}y_k} \to T_{x_k,y_k}$. We now have

$$\int_{T_{3^{N_k}x_k,3^{N_k}y_k}} |D\psi_k|^{4/3}dx = 3^{N_k} \int_{T_{x_k,y_k}} |D\psi_k|^{4/3}dx$$

and

$$\int_{T_{3^{N_k}x_k,3^{N_k}y_k}} \langle\psi_k, D\psi_k\rangle dx = 3^{N_k} \int_{T_{x_k,y_k}} \langle\psi_k, D\psi_k\rangle dx.$$

We then get by (14) that

$$C \le \lambda_{\min}^{3^{N_k}x^k,3^{N_k}y_k} \le J_{3^{N_k}(x_k,y_k)}(\psi_k) = 3^{N_k/2}\lambda_{\min}^{x_k,y_k} \le Cy_k^{-1/2}\lambda_{\min}^{x_k,y_k}.$$

Step 2. There exists $C > 0$ such that for all k, we have $\lambda_{\min}^{x_k,y_k} \ge C$.

Let $\eta : \mathbb{R} \to [0, 1]$ be a cut-off function defined on $\mathbb{R}$ which is equal to 0 on $\mathbb{R} \setminus [-1, 2]$ and which is equal to 1 on $[0, 1]$. We may assume that η is smooth. Let $v_k = (x_k, y_k)$. Since (e_1, v_k) is a basis of $\mathbb{R}^2$, we can define $\eta_k : \mathbb{R}^2 \to [0, 1]$ by

$$\eta_k(tv_k + se_1) = \eta(s)$$

Since v_k is asymptotically orthogonal to e_1, we can find $C > 0$ independent of k such that

$$|\nabla\eta_k| \le C \tag{15}$$

Moreover, by Corollary 4.4, we have

$$\frac{\left(\int_{Z_{x_k,y_k}} |D\eta_k\psi_k|^{4/3}dx\right)^{3/2}}{\left|\int_{Z_{x_k,y_k}} \langle\eta_k\psi_k, D\eta_k\psi_k\rangle dx\right|} \ge \lambda_{\min}(\mathbb{S}^2). \tag{16}$$

Now, we write that

$$\left(\int_{Z_{x_k,y_k}} |D\eta_k\psi_k|^{4/3}dx\right)^{3/4} = \left(\int_{Z_{x_k,y_k}} |\nabla\eta_k\psi_k + \eta_k D\psi_k|^{4/3}dx\right)^{3/4}$$
$$\le \left(\int_{Z_{x_k,y_k}} |\nabla\eta_k\psi_k|^{4/3}dx\right)^{3/4} + \left(\int_{Z_{x_k,y_k}} |\eta_k D\psi_k|^{4/3}dx\right)^{3/4}.$$

By (15) and Hölder inequality, we have

$$\left(\int_{Z_{x_k,y_k}} |\nabla\eta_k\psi_k|^{4/3}dx\right)^{3/4} \leq C\left(\int_{Z_{x_k,y_k}\cap\mathrm{Supp}(\nabla\eta_k)} |\psi_k|^{4/3}dx\right)^{3/4}$$
$$\leq C\left(\int_{Z_{x_k,y_k}\cap\mathrm{Supp}(\nabla\eta_k)} |\psi_k|^{4}dx\right)^{1/4} \mathrm{Vol}(Z_{x_k,y_k}\cap\mathrm{Supp}(\nabla\eta_k))^{1/2}.$$

We then have

$$\mathrm{Vol}(Z_{x_k,y_k}\cap\mathrm{Supp}(\nabla\eta_k)) \leq 3y_k.$$

By (12) and Step 1, this gives that

$$\left(\int_{Z_{x_k,y_k}} |\nabla\eta_k\psi_k|^{4/3}dx\right)^{3/4} \leq Cy_k^{1/2} \leq C\lambda_{\min}^{x_k,y_k}.$$

With the same argument and using relations (11) and (12), it follows that

$$\left(\int_{Z_{x_k,y_k}} |\eta_k D\psi_k|^{4/3}dx\right)^{3/4} \leq 3^{3/4}\lambda_{\min}^{x_k,y_k}\left(\int_{T_{x_k,y_k}} |\psi_k|^{4}dx\right)^{3/4} \leq C\lambda_{\min}^{x_k,y_k}.$$

Finally, we get that

$$\left(\int_{Z_{x_k,y_k}} |D\eta_k\psi_k|^{4/3}dx\right)^{3/2} \leq C\left(\lambda_{\min}^{x_k,y_k}\right)^2. \tag{17}$$

We now write that

$$\int_{Z_{x_k,y_k}} \langle\eta_k\psi_k, D\eta_k\psi_k\rangle dx = \int_{Z_{x_k,y_k}} \langle\eta_k\psi_k, \nabla\eta_k\psi_k + \eta_k D\psi_k\rangle dx.$$

Moreover, the left hand side of this equality is real since D is an autoadjoint operator. Since

$$\int_{Z_{x_k,y_k}} \langle\eta_k\psi_k, \nabla\eta_k\psi_k\rangle dx \in i\mathbb{R}.$$

Together with Eq. (11), this implies that

$$\int_{Z_{x_k,y_k}} \langle\eta_k\psi_k, D\eta_k\psi_k\rangle dx = \int_{Z_{x_k,y_k}} \eta_k^2\lambda_{\min}^{x_k,y_k}|\psi_k|^4 dx.$$

Using (12), we obtain that

$$\int_{Z_{x_k,y_k}} \langle \eta_k \psi_k, D\eta_k \psi_k \rangle dx \geq \lambda_{\min}^{x_k,y_k} \int_{T_{x_k,y_k}} |\psi_k|^4 dx = \lambda_{\min}^{x_k,y_k}. \tag{18}$$

Finally, plugging (17) and (18) in (16), we obtain that $\lambda_{\min}(\mathbb{S}^2) \leq C\lambda_{\min}^{x_k,y_k}$. This proves the step.

Step 3. The function $\lambda_{\min}$ can be extended continuously to $\mathcal{M}_1 \cup \{(0,0)\}$ by setting $\lambda_{\min}^{0,0} = \lambda_{\min}(\mathbb{S}^2)$.

In other words, we show that $\lim_k \lambda_{\min}^{x_k,y_k} = \lambda_{\min}(\mathbb{S}^2)$. The method is quite similar than the one of previous step. Let $\zeta_k : \mathbb{R} \to [0,1]$ be a smooth cut-off function defined on $\mathbb{R}$ which is equal to 0 on $\mathbb{R} \setminus [-y_k, 1+y_k]$, which is equal to 1 on $[0,1]$ and which satisfies $|\nabla \zeta_k| \leq 2/y_k$. As in the last step, we can define $\gamma_k : \mathbb{R}^2 \to [0,1]$ by

$$\gamma_k(tv_k + se_1) = \zeta_k(s).$$

Since v_k is asymptotically orthogonal to e_1, we can find $C > 0$ independent of k such that

$$|\gamma_k| \leq \frac{C}{y_k} \tag{19}$$

As in Step 2, we have

$$\frac{\left(\int_{Z_{x_k,y_k}} |D\gamma_k \psi_k|^{4/3} dx\right)^{3/2}}{\left|\int_{Z_{x_k,y_k}} \langle \gamma_k \psi_k, D\gamma_k \psi_k \rangle dx\right|} \geq \lambda_{\min}(\mathbb{S}^2). \tag{20}$$

We first prove that we can assume that

$$\int_{Z_{x_k,y_k} \cap \mathrm{Supp}(\nabla \gamma_k)} |\psi_k|^4 dx \leq C y_k. \tag{21}$$

We let $n_k = [(2y_k)^{-1}]$ be the integer part of $2{y_k}^{-1}$. For all $l \in [0, n_k - 1]$, we define

$$A_{k,l} = \left\{ te_1 + sv_k | s \in [0,1[\text{ and } t \in \left[\frac{l-(1/2)}{n_k}, \frac{l+(1/2)}{n_k}\right] \right\}.$$

The family of sets $(A_{k,l})_{l \in [0,n_k-1]}$ is a partition of T'_{x_k,y_k} which is the image of T_{x_k,y_k} by the translation of vector $-(1/2n_k)e_1$. By periodicity, $(A_{k,l})_{l \in [0,n_k-1]}$ can be seen as a partition of T_{x_k,y_k}. Consequently, we can write that

$$1 = \int_{T_{x_k,y_k}} |\psi_k|^4 dx = \sum_{l=0}^{n_k-1} \int_{A_{k,l}} |\psi_k|^4 dx.$$

Hence, there exists $l_0 \in [0, n_k - 1]$ such that

$$\int_{A_{k,l_0}} |\psi_k|^4 dx = \min_{l\in[0,n_k-1]} \sum_{l=0}^{n_k-1} \int_{A_{k,l}} |\psi_k|^4 dx \le \frac{1}{n_k}.$$

Obviously, without loss of generality, we can replace ψ_k by $\psi_k \circ t_0$ where t_0 is the translation of vector $-l_0 e_1$. In this way, we can assume that $l_0 = 0$. By periodicity, $\mathrm{Supp}(\nabla\gamma_k) \subset A_{k,0}$. Hence,

$$\int_{Z_{x_k,y_k}\cap \mathrm{Supp}(\nabla\gamma_k)} |\psi_k|^4 dx \le \frac{1}{n_k}.$$

Since $n_k \sim 2/y_k$, Eq. (21) follows.

Now, we proceed as in Step 2. We write that

$$\left(\int_{Z_{x_k,y_k}} |D\gamma_k\psi_k|^{4/3} dx\right)^{3/4} = \left(\int_{Z_{x_k,y_k}} |\nabla\gamma_k\psi_k + \gamma_k D\psi_k|^{4/3} dx\right)^{3/4}$$
$$\le \left(\int_{Z_{x_k,y_k}} |\nabla\gamma_k\psi_k|^{4/3} dx\right)^{3/4} + \left(\int_{Z_{x_k,y_k}} |\gamma_k D\psi_k|^{4/3} dx\right)^{3/4}.$$

It follows from (19) and the Hölder inequality that

$$\left(\int_{Z_{x_k,y_k}} |\nabla\gamma_k\psi_k|^{4/3} dx\right)^{3/4} \le \frac{C}{y_k}\left(\int_{Z_{x_k,y_k}\cap \mathrm{Supp}(\nabla\gamma_k)} |\psi_k|^{4/3} dx\right)^{3/4}$$
$$\le \frac{C}{y_k}\left(\int_{Z_{x_k,y_k}\cap \mathrm{Supp}(\nabla\gamma_k)} |\psi_k|^4 dx\right)^{1/4} (\mathrm{Vol}(Z_{x_k,y_k} \cap \mathrm{Supp}(\nabla\gamma_k)))^{1/2}.$$

Clearly, we have

$$\mathrm{Vol}(Z_{x_k,y_k} \cap \mathrm{Supp}(\nabla\gamma_k)) \le C y_k^2.$$

By (21), we obtain

$$\left(\int_{Z_{x_k,y_k}} |\nabla\gamma_k\psi_k|^{4/3} dx\right)^{3/4} \le C y_k^{-1+(1/4)+1} \le C y_k^{1/4} = o(1).$$

For the other term, we write, using (11)

$$\left(\int_{Z_{x_k,y_k}} |\gamma_k D\psi_k|^{4/3} dx\right)^{3/4} = \lambda_{\min}^{x_k,y_k}\left(\int_{T_{x_k,y_k}} |\psi_k|^4 dx + \int_{Z_{x_k,y_k}\cap\{0<\gamma_k<1\}} |\psi_k|^4 dx\right)^{3/4}.$$

Clearly, we can construct γ_k such that $\{0 < \gamma_k < 1\} \subset \mathrm{Supp}(\nabla\gamma_k)$. It then follows from (21) that

$$\left(\int_{Z_{x_k,y_k}} |\gamma_k D\psi_k|^{4/3} dx\right)^{3/4} \leq \lambda_{\min}^{x_k,y_k} + o(1).$$

Finally, we obtain

$$\left(\int_{Z_{x_k,y_k}} |D\gamma_k\psi_k|^{4/3} dx\right)^{3/2} \leq (\lambda_{\min}^{x_k,y_k})^2 + o(1). \tag{22}$$

Now, as in Step 2, we write that

$$\int_{Z_{x_k,y_k}} \langle\gamma_k\psi_k, D\gamma_k\psi_k\rangle dx = \int_{Z_{x_k,y_k}} \gamma_k^2\lambda_{\min}^{x_k,y_k}|\psi_k|^4 dx.$$

Using (12), we obtain that

$$\int_{Z_{x_k,y_k}} \langle\gamma_k\psi_k, D\gamma_k\psi_k\rangle dx \geq \lambda_{\min}^{x_k,y_k} \int_{T_{x_k,y_k}} |\psi_k|^4 dx = \lambda_{\min}^{x_k,y_k}. \tag{23}$$

Plugging (22) and (23) in (20), we obtain that

$$\lambda_{\min}(\mathbb{S}^2) \leq \frac{(\lambda_{\min}^{x_k,y_k})^2 + o(1)}{\lambda_{\min}^{x_k,y_k}}$$

which implies that either $\lambda_{\min}^{x_k,y_k} \to 0$ or $\lambda_{\min}^{x_k,y_k} \to \lambda_{\min}(\mathbb{S}^2)$. Hence, Step 2 yields the statement of the theorem. □

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